Mindlin Plate Finite Elements by a Modified Transverse Displacement

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The formulation of 4-node plate bending elements to eliminate the shear locking is presented. The kinematic variables in the Mindlin plate are related through the equilibrium equations; and then the transverse displacement, rotations, and shear strain are expressed in terms of the curvature sum. The elements are formulated by using the modified transverse displacement defined as the transverse displacement subtracted by the curvature sum times the ratio of flexural to shear rigidity. It is shown that the elements describe plate behavior quite correctly without any locking and also that they are applicable to the analysis of both thin and thick plates.

Key Words: Mindlin Plate, Shear Locking, Modified Transverse Displacement, Curvature Sum

1. Introduction

Over the decades considerable effort has been directed toward the development of improved plate and shell elements. The Mindlin theory (Mindlin, 1951) serves as the canonical starting point in the formulation of the conventional 'degenerated' structural elements. The elements seemed to have promising features from the computational viewpoints to the general analysis capabilities. It is, however, soon widely known that the elements have severe drawbacks when the structure becomes thin. Nowadays this highly undesirable situation of numerical difficulties in the elements is broadly referred to as 'locking' phenomenon (Bathe, 1982). Especially in the plate, the spurious shear strain energy actually not present causes poor behavior of the elements. Recent studies show that it is because the lower-order conventional elements cannot represent the condition of zero transverse shear strain when the elements are subjected to bending moment only (Bathe, 1982).

A wide range of researches have been conducted throughout the globe and several methods have been proposed to alleviate locking occurring in the elements since the phenomeon was first identified (Pawsey and Clough, 1971). The selective/reduced integration scheme (Pugh et al., 1978) may be the first used to overcome the numerical difficulties and yielding improved analysis results, although the overstiff solutions are still obtained in many cases. Other methods to deal with the locking include the discrete Kirchhoff theory, the shear penalty-parameter modification, the anisoparametric interpolations, the free formulation, the field consistency, the assumed-strain methods, and the hybrid/mixed models (see for survey, Hinton and Huang, 1986; Prathap and Babu, 1986; and Bathe et al., 1989).

All of the approaches have been successfully applied to make corrections for the deficiency of the behavior of the elements in the thin regime. A consensus, however, favoring any one particular approach is not yet in evidence. This is because some very special procedures are prevalently in use in the methods; the lack of mathematical consistency and generalities in some methods needs due consideration.

In this paper, a new and simple way of formulating the Mindlin element is proposed. The kinematic variables for both shear strains and

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curvatures are first related through the equilibrium equations. Then the elliptic type differential equation for the parameter representing the drilling degree of freedom is obtained. Based on the solution of this equation, the kinematic variables are interpreted and modified: the shear strains are represented by the differentiated form of the curvature sum and the other kinematic variables are also expressed in terms of it. The transverse displacement is modified to accomodate the shear effects properly. It is noted here that when the modified transverse displacement is replaced by the original nonmodified transverse displacement then all the equations are the same as in the Kirchhoff plate, and also that as the plate becomes very thin in the limiting case the modified term is coincident with the nonmodified one. A transformation matrix is finally introduced to relate the modified nodal displacement vector to the nonmodified one.

Two types of the 4-node Mindlin plate elements are formulated: one is the non-conforming plate element with 12 degrees of freedom and the other is the conforming plate element with 16 degrees of freedom. They are applied to several types of plate problems to verify the concepts employed and their capabilities of analysis. The solutions obtained reveal that the elements are capable of representing the behavior of the Mindlin plate quite well showing no locking phenomenon and that they are also applicable to the analysis of both thin and thick plates.

2. Self-Equilibrating Kinematic Variables

Before presenting the derivation of the 4-node plate element the equations of the Mindlin plate theory are briefly reviewed and a new parameter denoted as the modified transverse displacement is derived. It is shown here that the modified transverse displacement expressed in terms of the transverse displacement and curvature sum plays the same role as the transverse displacement in the Kirchhoff plate.

The main assumption of the Mindlin plate theory is that the normals to the plate midsurface

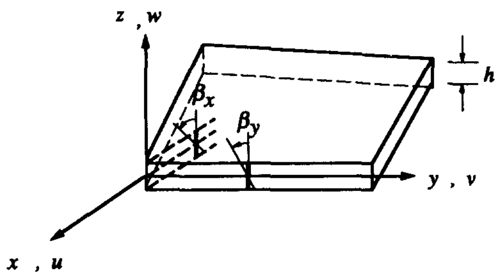


Fig. 1 Deformation assumptions in analysis of the Mindlin plate including shear effects

before deformation remain straight but not necessarily normal to the plate after deformation (Bathe, 1982). At a typical point in the Mindlin plate the displacement components in Fig. 1 are

$$u=z\beta_x(x,y),\ v=-z\beta_y(x,y),\ w=w(x,y), \quad (1)$$

where u and v are the displacement components in the x and y directions, respectively, w is the transverse displacement, and β_x and β_y are the rotations of the normal to the undeformed midsurface in the xz and yz planes, respectively.

The curvature-displacement relations are represented as:

$$\boldsymbol{\varepsilon}_b=\begin{bmatrix}\kappa_x\\ \kappa_y\\ \kappa_{xy}\end{bmatrix}=\begin{bmatrix}\beta_{x,x}\\ -\beta_{y,y}\\ \beta_{x,y}-\beta_{y,x}\end{bmatrix}, \quad (2)$$

and shear strain-displacement relations are written as

$$\boldsymbol{\varepsilon}_s=\begin{bmatrix}\gamma_{yz}\\ \gamma_{xz}\end{bmatrix}=\begin{bmatrix}w_{,y}-\beta_y\\ w_{,x}-\beta_x\end{bmatrix}. \quad (3)$$

The bending moment-curvature and shear force-shear strain relations are given respectively for a homogeneous isotropic plate as:

$$\boldsymbol{\sigma}_b=\boldsymbol{C}_b\boldsymbol{\varepsilon}_b, \quad (4a)$$

where

$$\boldsymbol{\sigma}_b=[M_x\ M_y\ M_{xy}]^T, \quad (4b)$$

$$\boldsymbol{C}_b=D_b\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix},\ D_b=\frac{Eh^3}{12(1-\nu^2)}, \quad (4c)$$

and

$$\boldsymbol{\sigma}_s=\boldsymbol{C}_s\boldsymbol{\varepsilon}_s, \quad (5a)$$

$$\boldsymbol{\sigma}_s=[V_{yz}\ V_{xz}]^T, \quad (5b)$$

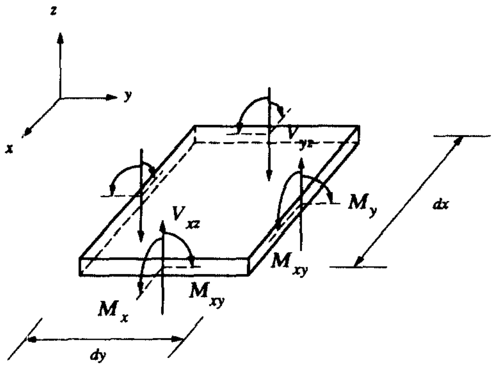


Fig. 2 Bending moments and shear forces acting on an infinitesimal element of the plate

$$C_s = D_s\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ D_s = Ghk. \tag{5c}$$

Here, h is the plate thickness and k is the shear correction factor usually taken as 5/6.

For an infinitesimally small element of the plate in Fig. 2, the following two equilibrium equations hold.

$$M_{x,x} + M_{xy,y} = V_{xz}, \tag{6a}$$

$$M_{xy,x} + M_{y,y} = V_{yz}. \tag{6b}$$

When the curvatures and shear strains in Eqs. (4) and (5) are inserted in Eq. (6), the shear strains can be represented by the curvatures as

$$\gamma_{yz} = \alpha(\nu\kappa_{x,y} + \kappa_{y,y} + \frac{1-\nu}{2}\kappa_{xy,x}), \tag{7a}$$

$$\gamma_{xz} = \alpha(\kappa_{x,x} + \nu\kappa_{y,x} + \frac{1-\nu}{2}\kappa_{xy,y}), \tag{7b}$$

where

$$\alpha = \frac{D_b}{D_s}. \tag{8}$$

Now, a new parameter which has the similar form to the twisting curvature κ_{xy} is introduced by

$$\bar{\omega} = \frac{1}{2}(\beta_{x,y} + \beta_{y,x}). \tag{9}$$

The above parameter is not a new one at all; it can be shown that the parameter represents the quantity of vorticity which is easily found, for example, in solid and fluid mechanics. Sometimes it is called the component of drilling degree of freedom. It will be referred to as vorticity hereafter.

The compatibility equation which holds for curvatures in the plate is as follows:

$$\kappa_{x,yy} + \kappa_{y,xx} = \kappa_{xy,xy}. \tag{10}$$

Rearranging the above equation results in the following expression.

$$\bar{\omega}_{,xy} = \bar{\omega}_{,yx}. \tag{11}$$

When the vorticity is expressed by the shear strains in Eq. (7) with the help of Eq. (2), then the following elliptic type differential equation is obtained (see Appendix).

$$\alpha\frac{(1-\nu)}{2}\nabla^2\bar{\omega} - \bar{\omega} = 0. \tag{12}$$

The vorticity is set to zero, which is equivalent to neglecting the rotation about the normal to the plate.

$$\bar{\omega} = 0. \tag{13}$$

The shape functions of plate elements usually in use are the polynomials in x and y. There is, however, no such polynomial solution that satisfies the elliptic differential Eq. (12) except a trivial one. It is, therefore, the simplest way to assume that the vorticity field is zero. Note that in most plate elements the nodal degrees of freedom for the vorticity is neglected.

With the assumption of Eq. (13), the shear strains in Eq. (7) can be finally expressed by the 'curvature sum' through a lengthy algebra (see Appendix for a full derivation) as

$$\boldsymbol{\varepsilon}_s = \begin{bmatrix}\gamma_{yz}\\ \gamma_{xz}\end{bmatrix} = \alpha\begin{bmatrix}\kappa_{,y}\\ \kappa_{,x}\end{bmatrix}, \tag{14}$$

where the curvature sum κ is

$$\kappa = \kappa_x + \kappa_y. \tag{15}$$

Hence the shear strain-displacement relations are then rewritten as

$$\begin{bmatrix}(w-\alpha\kappa)_{,y}\\ (w-\alpha\kappa)_{,x}\end{bmatrix} = \begin{bmatrix}\beta_y\\ -\beta_x\end{bmatrix}. \tag{16}$$

As seen in the equation above the transverse displacement is modified by the curvature sum multiplied by the ratio of flexural rigidity to shear rigidity. It will be referred to as the modified transverse displacement hereafter. In the limiting case of thin plate, the modified displacement becomes the same as the nonmodified one. To distinguish it from the original transverse displacement the capital letter will be used for it.

$$W = w - \alpha\kappa. \tag{17}$$

3. Finite Element Formulation

There are many ways to formulate plate elements when one is to use the basic equations for the kinematic variables derived in the previous section. Here in this paper, we present two types of 4-node plate elements, in which the modified transverse displacement field is first assumed to have interpolation functions used in the non-conforming 12 degree of freedom and in the conforming 16 degree of freedom plate element, respectively. The other kinematic parameters are obtained by differentiating it. The formulation of the Mindlin plate element from any Kirchhoff plate element is now possible in theory and the procedure is quite similar to that presented here.

By using a polynomial expression, the modified transverse displacement for the conforming 4-node element can be written as:

$$\begin{aligned} W &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 \\ &\quad + \alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3 \\ &\quad + \alpha_{11} x^3 y + \alpha_{12} xy^3 + \alpha_{13} x^2 y^2 + \alpha_{14} x^3 y^2 \\ &\quad + \alpha_{15} x^2 y^3 + \alpha_{16} x^3 y^3 \\ &\equiv \boldsymbol{P}\boldsymbol{a}. \end{aligned} \tag{18}$$

For the non-conforming 4-node plate element, however, the first 12 terms are used.

The constants α_1 to α_{16} (or α_{12}) can be evaluated by writing down the 16 (or 12) simultaneous equations for W_i, β_{xi}, β_{yi}, and $w_{,xyi}$ at the nodes ($w_{,xyi}$'s are absent for the non-conforming element case). Listing all these values in $\boldsymbol{V}$, we can write, in matrix form,

$$\boldsymbol{V} = \boldsymbol{A}\boldsymbol{a}, \tag{19}$$

where $\boldsymbol{A}$ is 16×16 (or 12×12) matrix depending on nodal coordinates and $\boldsymbol{a}$ is a vector of the 16 (or 12) unknown constants. Eq. (19), Inverting we have

$$\boldsymbol{a} = \boldsymbol{A}^{-1}\boldsymbol{V}. \tag{20}$$

Now it is possible to write the expressions for the variables in terms of the nodal values in a 4-node cubic element shown in Fig. 3 as

$$W = \boldsymbol{P}\boldsymbol{A}^{-1}\boldsymbol{V} = \boldsymbol{H}_W \boldsymbol{V}, \tag{21}$$

$$\beta_x = -\boldsymbol{P}_{,x}\boldsymbol{A}^{-1}\boldsymbol{V} = -\boldsymbol{H}_{W,x}\boldsymbol{V}, \tag{22}$$

$$\beta_y = \boldsymbol{P}_{,y}\boldsymbol{A}^{-1}\boldsymbol{V} = \boldsymbol{H}_{W,y}\boldsymbol{V}, \tag{23}$$

$$w_{,xy} = \boldsymbol{P}_{,xy}\boldsymbol{A}^{-1}\boldsymbol{V} = \boldsymbol{H}_{W,xy}\boldsymbol{V}. \tag{24}$$

The curvatures can be written from Eq. (2) by using the modified transverse displacement as

$$\varepsilon_b = \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = -\begin{bmatrix} W_{,xx} \\ W_{,yy} \\ 2W_{,xy} \end{bmatrix} = -\begin{bmatrix} \boldsymbol{H}_{W,xx} \\ \boldsymbol{H}_{W,yy} \\ 2\boldsymbol{H}_{W,xy} \end{bmatrix} \boldsymbol{V}. \tag{25}$$

The curvature sum is then given by

$$\kappa = \boldsymbol{H}_\kappa \boldsymbol{V}, \tag{26}$$

where

$$\boldsymbol{H}_\kappa = -(\boldsymbol{H}_{W,xx} + \boldsymbol{H}_{W,yy}). \tag{27}$$

The nodal vector $\boldsymbol{V}$ having the modified transverse displacements is transformed into the vector $\boldsymbol{U}$ in which the original transverse displacements are contained: i.e., the vector $\boldsymbol{U}$ is given by

$$\boldsymbol{U} = [w_1\ \beta_{x1}\ \beta_{y1}\ w_{,xy1}\ \cdots\ w_4\ \beta_{x4}\ \beta_{y4}\ w_{,xy4}]^T. \tag{28}$$

Again, $w_{,xyi}$'s are absent in $\boldsymbol{U}$ for the non-conforming element. From the definition of the modified transverse displacement, that is, Eq. (17), with sectional rotations, the following matrix equation holds.

$$\begin{bmatrix} W_1 \\ \beta_{x1} \\ \beta_{y1} \\ w_{,xy1} \\ \vdots \\ W_4 \\ \beta_{x4} \\ \beta_{y4} \\ w_{,xy4} \end{bmatrix} = \begin{bmatrix} w_1 \\ \beta_{x1} \\ \beta_{y1} \\ w_{,xy1} \\ \vdots \\ w_4 \\ \beta_{x4} \\ \beta_{y4} \\ w_{,xy4} \end{bmatrix} - \alpha \begin{bmatrix} \boldsymbol{H}_\kappa|_1 \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{H}_\kappa|_4 \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{V}. \tag{29}$$

In a concise matrix form,

$$\boldsymbol{V} = \boldsymbol{U} - \alpha \boldsymbol{H}_\kappa|_n \boldsymbol{V} \quad \text{or} \quad \boldsymbol{V} = \boldsymbol{T}\boldsymbol{U}, \tag{30}$$

where

$$\boldsymbol{T} = [\boldsymbol{I} + \alpha \boldsymbol{H}_\kappa|_n]^{-1}. \tag{31}$$

In the earlier section, it is shown that the shear strains are obtained by differentiating the curvature sum. From Eq. (25), we have

$$\boldsymbol{\varepsilon}_s = \begin{bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{bmatrix}$$

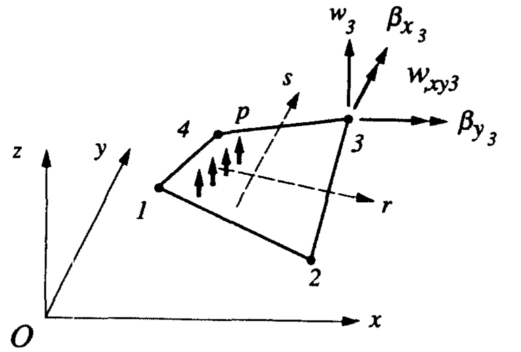


Fig. 3 Plate bending element with transverse loads applied : The nodal degrees of freedom are transverse displacement and two sectional rotations

$$=\alpha\begin{bmatrix} K_{,y} \\ K_{,x} \end{bmatrix}=\alpha\begin{bmatrix} \boldsymbol{H}_{x,x} \\ \boldsymbol{H}_{x,y} \end{bmatrix}\boldsymbol{V}$$

$$=\alpha\begin{bmatrix} -(\boldsymbol{H}_{W,xxx}+\boldsymbol{H}_{W,yyx}) \\ -(\boldsymbol{H}_{W,xxy}+\boldsymbol{H}_{W,yyy}) \end{bmatrix}\boldsymbol{V}. \qquad (32)$$

The element equilibrium equations are established by invoking the stationarity condition of the total potential energy. The total potential energy for a single plate element in Fig. 3 is

$$\pi=\frac{1}{2}\int_A \boldsymbol{\varepsilon}_b^T C_b \boldsymbol{\varepsilon}_b dA+\frac{1}{2}\int_A \boldsymbol{\varepsilon}_s^T C_s \boldsymbol{\varepsilon}_s dA$$

$$-\int_A wpdA, \qquad (33)$$

where p is the transverse load per unit area. The stationarity condition of $\delta\pi=0$ with all the work done for the kinematic variables and strains, the final finite element equilibrium equation is obtained for a single element as

$$\boldsymbol{KU}=\boldsymbol{R} \qquad (34)$$

in which the stiffness matrix is

$$\boldsymbol{K}=\boldsymbol{T}^T\Big(\int_A[\boldsymbol{H}_{W,xx}^T\ \boldsymbol{H}_{W,yy}^T\ 2\boldsymbol{H}_{W,xy}^T]C_b\begin{bmatrix} \boldsymbol{H}_{W,xx} \\ \boldsymbol{H}_{W,yy} \\ 2\boldsymbol{H}_{W,xy} \end{bmatrix}dxdy$$

$$+\alpha^2\int_A[\boldsymbol{H}_{x,x}^T\ \boldsymbol{H}_{x,y}^T]C_s\begin{bmatrix} \boldsymbol{H}_{x,x} \\ \boldsymbol{H}_{x,y} \end{bmatrix}dxdy\Big)\boldsymbol{T}, \qquad (35)$$

and the load vector is

$$\boldsymbol{R}=\boldsymbol{T}^T\Big(\int_A(\boldsymbol{H}_W+a\boldsymbol{H}_x)^T p dxdy\Big). \qquad (36)$$

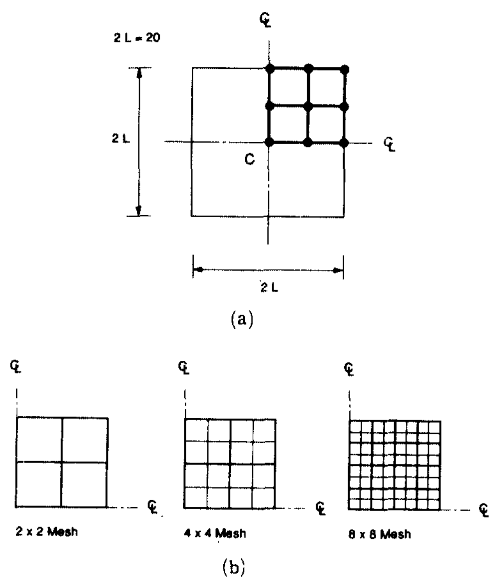
4. Numerical Examples

Numerical examples are now presented to test the plate elements just formulated in section 3. Square plate problems are solved by using the non-conforming 12 degree of freedom and conforming 16 degree of freedom 4-node elements. Material constants and geometric dimensions used are as follows: Poisson's ratio $=0.3$, Young's modulus $=1.092\times10^{11}$ Pa, length L $=20$ cm.

4.1 Analysis of a square plate

This set of problems is the most commonly employed one for testing the plate element behavior. Regular meshes and boundary conditions for a quarter of the plate used in the analysis are depicted in Figs. 4(a) and (b).

Tables 1 and 2 summarize the results of the maximum transverse displacement at the center of the plate for the non-conforming 12 degree of freedom and the conforming 16 degree of freedom 4-node plate element, respectively. In the tables, the transverse displacements are normalized with respect to the analytic Kirchhoff solution (Timoshenko and Woinowsky-Krieger, 1959). From these results, we see the high predictive capability of the elements in the analysis of the square plate with no locking.



(a) one quarter of plate considered due to its symmetry
(b) Meshes used

Fig. 4 Analysis of a square plate

Table 1 Normalized transverse displacement at the center of the square plate with the non-conforming 4-node plate elements with 12 degrees of freedom

(a) Effect of plate thickness for 2×2 mesh

Thickness	Concentrated load		Uniform pressure	
	Simply supported	Clamped	Simply supported	Clamped
2	1.165	1.316	1.114	1.280
0.2	1.073	1.099	1.068	1.116
0.02	1.072	1.097	1.067	1.114
0.002	1.072	1.097	1.067	1.114

(b) Effect of mesh density for thickness=0.02

Mesh density	Concentrated load		Uniform pressure	
	Simply supported	Clamped	Simply supported	Clamped
2×2	1.072	1.097	1.067	1.114
4×4	1.020	1.037	1.018	1.035
8×8	1.006	1.014	1.006	1.012

Table 2 Normalized transverse displacement at the center of the square plate with the conform-ing 4-node plate elements with 16 degrees of freedom

(a) Effect of plate thickness for 2×2 mesh

Thickness	Concentrated load		Uniform pressure	
	Simply supported	Clamped	Simply supported	Clamped
2	0.925	1.076	0.873	0.046
0.2	0.932	0.943	0.939	0.965
0.02	0.930	0.938	0.938	0.961
0.002	0.930	0.938	0.938	0.961

(b) Effect of mesh density for thickness=0.02

Mesh density	Concentrated load		Uniform pressure	
	Simply supported	Clamped	Simply supported	Clamped
2×2	0.930	0.938	0.938	0.961
4×4	0.949	0.965	0.950	0.973
8×8	0.955	0.975	0.953	0.980

4.2 Square plate subjected to distributed edge moments

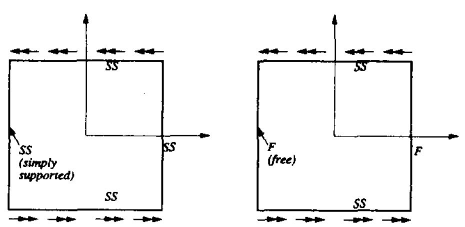
A square plate subjected to symmetrically distributed uniform edge moments on the two simply supported opposite edges shown in Fig. 5 is tested. Two different boundary conditions as (a) and (b) in the figure are considered : the other two edges are simply supported in (a) ; and free in (b).

In Tables 3(a) and 3(b), the values of the normalized center transverse displacements are given. The displacements are again normalized with respect to the Kirchhoff solutions (Timoshenko and Woinowsky-Krieger,1959; and Kang, 1992). The meshes are shown in Fig. 6.

4.3 Twisting of a square plate

As shown in Fig. 7, four corner forces are applied to a square plate whose four edges are free. Due to symmetry, one quarter of the plate is analyzed with the proper boundary conditions imposed.

In Table 4, the normalized corner transverse displacements are presented ; the nomalization is again done with respect to the Kirchhoff solution (Kang,1992). It shows that the displacement solutions are very accurate regardless of the types of the element and mesh used, except for the very thick case. As expected, the values of $w_{,xy}$, and



(a) (b)

(a) four edges are simply suppoprted
(b) two edges are simply supported and the other two are free

Fig. 5 Bending of a square plate

Table 3 Normalized transverse displacement at the center of the square plate subjected to sym-metrically distributed uniform edge moments on the two simply supported opposite edges

(a) When the other two edges are simply supported :

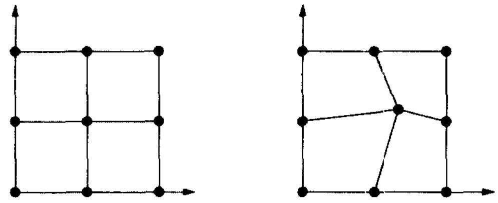
Thickness	16 d.o.f. element		12 d.o.f. element	
	Square mesh	Distorted mesh	Square mesh	Distorted mesh
2	0.994	1.058	1.054	1.146
0.2	1.008	1.131	1.064	1.181
0.02	1.008	1.131	1.064	1.181
0.002	1.008	1.131	1.064	1.181

(b) When the other two edges are free :

Thickness	16 d.o.f. element		12 d.o.f. element	
	Square mesh	Distorted mesh	Square mesh	Distorted mesh
2	0.954	0.933	0.853	0.871
0.2	1.023	0.989	0.860	0.956
0.02	1.023	0.989	0.860	0.956
0.002	1.023	0.990	0.860	0.957

thus those of twisting moment are constant with the square mesh, but not with the distorted mesh for both elements.

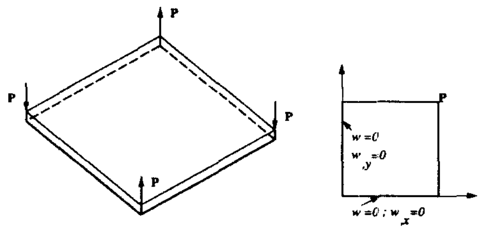
The results of the above examples show that the Mindlin plate elements formulated from the existing Kirchhoff plate elements are excellent in dealing with the locking problem. There is, however, one point worth mentioning: the behavior of the resulting Mindlin plate element formulated through the procedure from any Kirchhoff plate element is expected to be the same as that of the original Kirchhoff plate element. The patch test is one example: the resulting Mindlin element can pass the test only if the original Kirchhoff element can. As is already well reported (for example, Zienkiewicz and Taylor, 1991), the non-conforming element can pass the patch test when its shape is rectangle or parallelogram. Therefore, the distorted mesh of the present Mindlin element of this



(a) (b)

(a) square mesh, and
(b) distorted mesh

Fig. 6 Meshes for the constant moment and twist cases



(a) (b)

(a) a square plate with four free edges subjected to four corner forces
(b) one quarter model of the plate analyzed

Fig. 7 Twisting of a square plate

Table 4 Normalized transverse displacement at the center of the square plate subjected four corner forces

Thickness	16 d.o.f. element		12 d.o.f. element	
	Square mesh	Distorted mesh	Square mesh	Distorted mesh
2	1.056	1.152	1.034	1.067
0.2	1.000	1.029	1.000	1.009
0.02	1.000	1.028	1.000	1.009
0.002	1.000	1.028	1.000	1.009

kind cannot capture the states of constant curvature. Some of the results showing about 18 percent error in Table 3 for the case of symmetrically distributed uniform edge moments might be due to this.

The geometry of the conforming 16 degree of freedom element must be rectangular with sides parallel to the global system of reference. This is when the element is formulated using the Hermitian interpolation functions (Dhatt and Touzot,1984; and Zienkiewicz and Taylor,1991). The present element of this type is, however, based on the generalized coordinate model, and therefore the restriction is not applicable. Tables 3 and 4 show that the distorted mesh can produce acceptably good results although, in some cases, the error is about 13 percent.

5. Concluding Remarks

In this paper, a new way of formulating the Mindlin plate element is proposed and two types of 4-node plate elements are presented to remedy the undesirable shear locking phenomenon. Numerical experiments conducted for the several problems show that the proposed elements predict the behavior of the Mindlin plate quite excellently without locking. It is also found that the elements can handle both thin and thick plates.

The basic mechanics interpreted through the consideration of equilibrium equations in the formulation can be used for the development of locking-free and enhanced plate elements. The two types of the 4-node plate elements considered here are the simplest ones; they are chosen only to demonstrate how the Mindlin plate element can be formulated from the Kirchhoff plate element as described in section 2. Considerable further work thus remains to be done not only in exploring the behavior of the elements herein but also in developing new and better elements.

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Appendix

Considering Eqs. (3) and (10) yields

$$\frac{1}{2}(-\gamma_{yz,x}+\gamma_{xz,y})=\frac{1}{2}(\beta_{x,y}+\beta_{y,x})=\bar{\omega}. \qquad (A.1)$$

The shear strains in Eq. (7) are expressed by rotations using Eq. (2) as

$$\gamma_{yz}=\alpha\left[\frac{1+\nu}{2}\beta_{x,xy}-\frac{1-\nu}{2}\beta_{y,xx}-\beta_{y,yy}\right], \qquad (A.2a)$$

$$\gamma_{xz}=\alpha\left[\beta_{x,xx}+\frac{1-\nu}{2}\beta_{x,yy}-\frac{1-\nu}{2}\beta_{y,xy}\right]. \quad \text{(A.2b)}$$

Differentiating each shear strain with respect to x and y, respectively becomes

$$\frac{1}{2}(-\gamma_{yz,x}+\gamma_{xz,y})=\alpha\frac{1-\nu}{2}\nabla^2\bar{\omega}. \quad \text{(A.3)}$$

Then, the elliptic type differential equation for vorticity is obtained from Eqs. (A.1) and (A.3).

$$\alpha\frac{1-\nu}{2}\nabla^2\bar{\omega}-\bar{\omega}=0. \quad \text{(A.4)}$$

The vorticity is assumed to be zero as

$$\bar{\omega}=0. \quad \text{(A.5)}$$

Now, the shear strains are rewritten with the assumption of (A.5) as

$$\begin{aligned}\gamma_{yz}&=\alpha\left[\frac{1+\nu}{2}\beta_{x,xy}+\left(\frac{1+\nu}{2}\beta_{y,xx}-\nabla^2\beta_y\right)\right]\\&=\alpha\left[\frac{1+\nu}{2}(\beta_{x,y}+\beta_{y,x})_{,x}-\nabla^2\beta_y\right]\\&=-\alpha\nabla^2\beta_y,\end{aligned} \quad \text{(A.6a)}$$

and similarly,

$$\gamma_{xz}=\alpha\nabla^2\beta_x. \quad \text{(A.6b)}$$

The definition of the curvature and the condition of (A.5) together yield the following relations.

$$\begin{aligned}\nabla^2(\beta_x+\beta_y)&=\beta_{x,xx}+\beta_{x,yy}+\beta_{y,xx}+\beta_{y,yy}\\&=\beta_{x,xx}-(\beta_{y,x})_{,y}-(\beta_{x,y})_{,x}+\beta_{y,yy}\\&=(\beta_{x,x}-\beta_{y,y})_{,x}-(\beta_{x,x}-\beta_{y,y})_{,y}\\&=(\kappa_x+\kappa_y)_{,x}-(\kappa_x+\kappa_y)_{,y},\end{aligned} \quad \text{(A.7a)}$$

and similarly

$$\nabla^2(\beta_x-\beta_y)=(\kappa_x+\kappa_y)_{,x}+(\kappa_x+\kappa_y)_{,y}. \quad \text{(A.7b)}$$

Considering Eqs. (A.6), (A.7) and (3) together gives

$$\begin{aligned}&\alpha[(\kappa_x+\kappa_y)_{,x}-(\kappa_x+\kappa_y)_{,y}]\\&=(\beta_x+\beta_y)+(w_{,x}-w_{,y}),\end{aligned} \quad \text{(A.8a)}$$

$$\begin{aligned}&\alpha[(\kappa_x+\kappa_y)_{,x}+(\kappa_x+\kappa_y)_{,y}]\\&=(\beta_x-\beta_y)+(w_{,x}+w_{,y}).\end{aligned} \quad \text{(A.8b)}$$

and further yields the following relations.

$$w_{,x}+\beta_x=\alpha(\kappa_x+\kappa_y)_{,x}, \quad \text{(A.9)}$$

$$w_{,y}-\beta_y=\alpha(\kappa_x+\kappa_y)_{,y}. \quad \text{(A.10)}$$

The left hand sides of these equations are the shear strains in Eq. (3) and hence

$$\begin{bmatrix}\gamma_{yz}\\ \gamma_{xz}\end{bmatrix}=\alpha\begin{bmatrix}\kappa_{,y}\\ \kappa_{,x}\end{bmatrix}, \quad \text{(A.11)}$$

where the curvature sum is

$$\kappa=\kappa_x+\kappa_y. \quad \text{(A.12)}$$

Finally, Eqs. (A.9) and (A.12) give the modified transverse displacement field as

$$W=w-\alpha\kappa. \quad \text{(A.13)}$$

That is, the modified transverse displacement is the original transverse displacement minus the curvature sum multiplied by the ratio of flexural to shear rigidity. Note that when the modified term is used all the relevant equations look exactly the same except W appearing in the Mindlin plate in place of w in the Kirchhoff plate.